

LIDSTONE POLYNOMIALS AND BOUNDARY VALUE PROBLEMS

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Abstract—We define Lidstone polynomials, provide their explicit representations, give their relations with Bernoulli polynomials, obtain best possible error estimates in Lidstone interpolation, and establish several best possible/sharp inequalities which compare favourably with the known results. Next, we use these results to study even order differential equations together with Lidstone boundary conditions.

1. INTRODUCTION

In the year 1929 Lidstone [1] introduced a generalization of Taylor's series, it approximates a given function in the neighborhood of two points instead of one. From practical points of view such a development is very useful and it is completely characterized in the papers of Boas [2, 3], Poritsky [4], Schoenberg [5], Whittaker [6], Widder [7, 8] and others. In the field of approximation theory for a given function $x(t) \in C^{(2m)}[0, 1]$, the Lidstone interpolating polynomial [9] of degree $(2m - 1)$ matches $x(t)$ and its $(m - 1)$ even derivatives $x^{(2i)}(t)$, $0 \leq i \leq m - 1$ at 0 and 1. Since this interpolating polynomial is a solution of the simplest differential equation $x^{(2m)} = 0$, the general $2m$ th nonlinear differential equation

$$(-1)^m x^{(2m)} = f(t, x, x', \dots, x^{(q)}) \quad (1)$$

together with the boundary conditions

$$x^{(2i)}(0) = \alpha_i, \quad x^{(2i)}(1) = \beta_i; \quad i = 0, 1, \dots, m - 1 \quad (2)$$

gives the possibility of interpolation by the solutions of the differential equation (1).

In the differential equation (1) we shall assume that $0 \leq q \leq 2m - 1$ but fixed, and the function f is at least continuous in its arguments.

Besides in approximation theory, the particular case $m = 2$, $f = f(t)x + g(t)$ of the boundary value problem (1), (2) frequently occurs in engineering and other branches of physical sciences. For instance, the deflection of a uniformly loaded rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges leads to this type of problem, e.g. see Agarwal [10], Aslam Noor and Tirmizi [11], Timoshenko and Kreiger [12], Usmani [13] and the several references therein. Motivated with the practical applications of the particular cases, the general case (1), (2) with $q = 0$ also has been considered by Agarwal and Akrivis [14], and Chawla and Katti [15].

The plan of this paper is as follows: In Section 2, we have a series of 15 lemmas and 16 following remarks. In the first 13 lemmas and following remarks we define Lidstone polynomials, provide their explicit representations, give their relations with the Bernoulli polynomials, obtain best possible error estimates in Lidstone interpolation, and establish several inequalities some of which are needed later whereas others compare sharply with the several supporting results of Boas [3] and Widder [7, 8] and hence are of independent interest. As a consequence of these inequalities we improve estimates of Jordan [16] for the Euler polynomials and Euler numbers. The last two lemmas are devoted to fixed point results which are needed later.

The results of Section 2 are used in Section 3 to derive necessary and sufficient conditions for the existence and uniqueness of the solutions of the boundary value problem (1), (2). These results are sharper than those known for the particular cases in [10, 14]. In Section 4, we provide an *a priori* as well as *posteriori* estimates on the Lipschitz constants so that the Picard iterative sequence

$\{x_n(t)\}$ generated by the iterative scheme (65) converges to a unique solution $x^*(t)$ of (1), (2). For the particular case $q = 0$, these estimates are the best possible. In practical evaluation of Picard's iterative sequence $\{x_n(t)\}$ only an approximate sequence $\{y_n(t)\}$ is constructed and this depends on approximating f by some simpler function. In Section 5, to find $y_{n+1}(t)$ we approximate f by f_n by following relative and absolute error criterion and provide necessary and sufficient conditions for the convergence of $\{y_n(t)\}$ to the solution $x^*(t)$ of (1), (2). Finally, in Section 6 the method of monotonic convergence which has attracted remarkable attention in the last ten years is used to construct the multiple solutions of (1), (2) with $q = 0$. Finally, we note that for the problem (1), (2) shooting type methods proposed in [10] can be used directly or after converting it into its equivalent first order systems.

2. SOME BASIC LEMMAS AND REMARKS

We begin with the following fundamental result due to Widder [8].

Lemma 2.1

Let $x(t) \in C^{(2m)} [0, 1]$, then

$$x(t) = \sum_{k=0}^{m-1} [x^{(2k)}(0)A_k(1-t) + x^{(2k)}(1)A_k(t)] + \int_0^1 G_m(t, s)x^{(2m)}(s) ds \quad (3)$$

where

$$G_1(t, s) = G(t, s) = \begin{cases} (t-1)s, & s \leq t, \\ (s-1)t, & t \leq s, \end{cases} \quad (4)$$

$$G_n(t, s) = \int_0^1 G(t, t_1)G_{n-1}(t_1, s) dt_1; \quad n = 2, 3, \dots \quad (5)$$

and $A_n(t)$ is the unique polynomial (Lidstone polynomial) of degree $(2n+1)$ defined by the relations

$$\begin{aligned} A_0(t) &= t, \\ A_n''(t) &= A_{n-1}(t), \\ A_n(0) &= A_n(1); \quad n = 1, 2, \dots \end{aligned} \quad (6)$$

which is in terms of $G_n(t, s)$ and the Fourier series expansion can be expressed as

$$A_n(t) = \int_0^1 G_n(t, s)s ds = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t. \quad (7)$$

Remark 2.1

It is well known (e.g. see Luke [17, p. 23]) that the Bernoulli polynomials may be expressed as

$$B_{2n+1}(t) = (-1)^{n+1} \frac{2}{(2\pi)^{2n+1}} (2n+1)! \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \sin 2k\pi t \quad (8)$$

and hence

$$\frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{-1}{k^{2n+1}} \sin(k\pi + k\pi t),$$

i.e. it follows that

$$A_n(t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{1+t}{2}\right). \quad (9)$$

Relation (9) between Lidstone and Bernoulli polynomials was earlier observed by Whittaker [6].

Remark 2.2

Another explicit representation of Lidstone polynomials is given by

$$A_n(t) = \frac{1}{6} \left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!} \right] - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}; \quad n = 1, 2, \dots \quad (10)$$

where B_{2k+4} is the $(2k+4)$ th Bernoulli number. The proof of (10) is by induction. Indeed, for $n = 1$ the relation (7) gives

$$A_1(t) = \int_0^1 G_1(t, s) s \, ds = \int_0^t (t-1)s^2 \, ds + \int_t^1 t(s^2-s) \, ds = \frac{1}{6}t^3 - \frac{1}{6}t$$

which is same as (10) for $n = 1$. Now assuming (10) to be true for n , from (7) it follows that

$$\begin{aligned} \frac{1}{6} \left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!} \right] - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!} \\ = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t. \end{aligned} \quad (11)$$

Integrating (11) twice from 0 to t , we obtain

$$\begin{aligned} \frac{1}{6} \left[\frac{6t^{2n+3}}{(2n+3)!} - \frac{t^{2n+1}}{(2n+1)!} \right] - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-1}}{(2n-2k-1)!} \\ = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \left[\frac{1}{k\pi} \left(t - \frac{1}{k\pi} \sin k\pi t \right) \right] \\ = (-1)^{n+1} \frac{2}{\pi^{2n+3}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+3}} \sin k\pi t + (-1)^n \frac{2}{\pi^{2n+2}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+2}} t \\ = A_{n+1}(t) - (-1)^{n+1} \frac{2}{\pi^{2n+2}} \frac{(2^{2n+1}-1)}{(2n+2)!} \pi^{2n+2} |B_{2n+2}| t, \end{aligned} \quad (12)$$

where in (12) we have used another well known (Jordan [16, p. 244]) relation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+2}} = \frac{(2^{2n+1}-1)}{(2n+2)!} \pi^{2n+2} |B_{2n+2}|. \quad (13)$$

However, since $(-1)^n B_{2n+2} > 0$ from (12) the relation (10) for $n+1$ follows immediately.

Remark 2.3

Equating (9) and (10), we get the polynomial expansion of $B_{2n+1}((1+t)/2)$. Further, as a consequence of this equality it directly follows that $B_{2n+1}(\frac{1}{2}) = 0$.

Remark 2.4

Widder [8] has shown that

$$A_n(1-t) = \int_0^1 G_n(t, s)(1-s) \, ds. \quad (14)$$

We can find an explicit representation of $A_n(1-t)$ from (9). Indeed, we have

$$A_n(1-t) = \frac{2^{2n+1}}{(2n+1)!} B_{2n+1} \left(1 - \frac{t}{2} \right).$$

However, for the Bernoulli polynomials it is known (Luke [17, p. 20]) that $B_{2n+1}(1-t/2) = -B_{2n+1}(t/2)$, and hence

$$\begin{aligned} A_n(1-t) &= -\frac{2^{2n+1}}{(2n+1)!} B_{2n+1}\left(\frac{t}{2}\right) \\ &= -\frac{2^{2n+1}}{(2n+1)!} \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix} \left(\frac{t}{2}\right)^k B_{2n+1-k}. \end{aligned} \quad (15)$$

The second interesting result of Widder [8] is stated in

Lemma 2.2

The following equality holds:

$$\int_0^1 G_n(t, s) ds = (-1)^n \frac{4}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi t}{(2k+1)^{2n+1}}; \quad n = 1, 2, \dots \quad (16)$$

Remark 2.5

Lindelöf has shown that the Euler polynomials $E_{2n}(t)$ can be expanded in terms of Fourier series and the following equality holds (see Jordan [16, p. 294]):

$$E_{2n}(t) = (-1)^n \frac{4}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi t}{(2k+1)^{2n+1}}. \quad (17)$$

Thus, from (16) and (17) it follows that

$$\int_0^1 G_n(t, s) ds = E_{2n}(t); \quad n = 1, 2, \dots \quad (18)$$

Further, from (4) we have $G_1(t, s) \leq 0$ for all $0 \leq s, t \leq 1$ and hence from (5) and (18) we immediately obtain that $(-1)^n E_{2n}(t) \geq 0$.

Remark 2.6

Since

$$A_n(1-t) = \int_0^1 G_n(t, s)(1-s) ds = \int_0^1 G_n(t, s) ds - \int_0^1 G_n(t, s)s ds$$

from (7) and (8) it follows that

$$E_{2n}(t) = A_n(t) + A_n(1-t) \quad (19)$$

and hence from (9) and (14), we have

$$E_{2n}(t) = \frac{2^{2n+1}}{(2n+1)!} \left[B_{2n+1}\left(\frac{1+t}{2}\right) - B_{2n+1}\left(\frac{t}{2}\right) \right]. \quad (20)$$

Fort [18, p. 41] has defined Euler's polynomials as we have here in (20).

Remark 2.7

As in Remark 2.2 by induction we shall show that

$$\int_0^1 G_n(t, s) ds = \frac{1}{2} \left(\frac{2t^{2n}}{(2n)!} - \frac{t^{2n-1}}{(2n-1)!} \right) - \sum_{k=0}^{n-2} \frac{2(2^{2k+4}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}; \quad n = 1, 2, \dots \quad (21)$$

For $n = 1$, by direct computation we have

$$\int_0^1 G_1(t, s) ds = \frac{1}{2}(t^2 - t)$$

which is same as (21) for $n = 1$. Now assuming (21) to be true for n , from (16) and (21) we have

$$\frac{1}{2} \left(\frac{2t^{2n}}{(2n)!} - \frac{t^{2n-1}}{(2n-1)!} \right) - \sum_{k=0}^{n-2} \frac{2(2^{2k+4}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!} = (-1)^n \frac{4}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi t}{(2k+1)^{2n+1}}. \quad (22)$$

Now integrating (22) twice from 0 to t , we find

$$\begin{aligned} & \frac{1}{2} \left(\frac{2t^{2n+2}}{(2n+2)!} - \frac{t^{2n+1}}{(2n+1)!} \right) - \sum_{k=0}^{n-2} \frac{2(2^{2k+4}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-1}}{(2n-2k-1)!} \\ &= (-1)^n \frac{4}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}} \left[\frac{1}{(2k+1)\pi} \left(t - \frac{1}{(2k+1)\pi} \sin(2k+1)\pi t \right) \right] \\ &= (-1)^{n+1} \frac{4}{\pi^{2n+3}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi t}{(2k+1)^{2n+3}} + (-1)^n \frac{4}{\pi^{2n+2}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} t \\ &= \int_0^1 G_{n+1}(t, s) ds + \frac{2(2^{2n+2}-1)}{(2n+2)!} B_{2n+2} t, \end{aligned} \quad (23)$$

where in (23) we have used the relation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+2}} = \frac{(2^{2n+2}-1)}{2(2n+2)!} (-1)^n \pi^{2n+2} B_{2n+2}, \quad (24)$$

which is a direct consequence of the known (Milne-Thomson [19, p. 138]) relation

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n+2}} = (-1)^n \frac{(2\pi)^{2n+2}}{2(2n+2)!} B_{2n+2}. \quad (25)$$

Equation (23) is same as (21) for $n+1$.

Lemma 2.3

The following holds:

$$\int_0^1 (-1)^n G_n(t, s) ds = \int_0^1 |G_n(t, s)| ds = (-1)^n E_{2n}(t) \leq (-1)^n E_{2n}(\tfrac{1}{2}) = \frac{(-1)^n E_{2n}}{2^{2n}(2n)!} \quad (26)$$

where E_{2n} is the $2n$ th Euler number.

Proof. Since the extreme of the function $E_{2n}(t)$ is at $t = \frac{1}{2}$ (Jordan [16, p. 293]) Lemma 2.3 is obvious.

Remark 2.8

From (3)–(6), we successively have

$$\begin{aligned} x^{(2i)}(t) &= \sum_{k=i}^{m-1} [x^{(2k)}(0) A_k^{(2i)}(1-t) + x^{(2k)}(1) A_k^{(2i)}(t)] + \int_0^1 G_{m-i}(t, s) x^{(2m)}(s) ds \\ &= \sum_{k=i}^{m-1} [x^{(2k)}(0) A_{k-i}(1-t) + x^{(2k)}(1) A_{k-i}(t)] + \int_0^1 G_{m-i}(t, s) x^{(2m)}(s) ds \\ &= \sum_{k=0}^{m-i-1} [x^{(2k+2i)}(0) A_k(1-t) + x^{(2k+2i)}(1) A_k(t)] + \int_0^1 G_{m-i}(t, s) x^{(2m)}(s) ds \end{aligned}$$

and now Lemma 2.3 implies that

$$\left| x^{(2i)}(t) - \sum_{k=0}^{m-i-1} [x^{(2k+2i)}(0)A_k(1-t) + x^{(2k+2i)}(1)A_k(t)] \right| \\ \leq (-1)^{m-i} E_{2m-2i}(t) M_{2m} \leq \frac{(-1)^{m-i} E_{2m-2i}}{2^{2m-2i} (2m-2i)!} M_{2m}; \quad i = 0, 1, \dots, m-1, \quad (27)$$

where

$$M_{2m} = \max_{0 \leq t \leq 1} |x^{(2m)}(t)|.$$

Inequalities (27) are obviously the best possible. A similar observation also appears in a recent publication of Varma and Howell [20].

Lemma 2.4

The following holds:

$$\int_0^1 |G'_n(t, s)| \, ds = (-1)^n [2E_{2n}(t) + (1-2t)E_{2n-1}(t)] \quad (28)$$

$$\leq (-1)^{n+1} \frac{2(2^{2n}-1)}{(2n)!} B_{2n}. \quad (29)$$

Proof. From (5), we have

$$\int_0^1 |G'_n(t, s)| \, ds = \int_0^1 \int_0^t t_1 (-1)^{n-1} G_{n-1}(t_1, s) \, dt_1 \, ds + \int_0^1 \int_t^1 (1-t_1) (-1)^{n-1} G_{n-1}(t_1, s) \, dt_1 \, ds$$

which is on changing the order of integration and (26) is same as

$$\int_0^1 |G'_n(t, s)| \, ds = (-1)^{n-1} \left(\int_0^t t_1 E_{2n-2}(t_1) \, dt_1 + \int_t^1 (1-t_1) E_{2n-2}(t_1) \, dt_1 \right). \quad (30)$$

Now in (30), we use the relation $E'_k(t) = E_{k-1}(t)$ (Jordan [16, p. 288]) and the conditions $E_{2n}(0) = E_{2n}(1) = 0$, to obtain

$$\int_0^1 |G'_n(t, s)| \, ds = (-1)^{n-1} [tE_{2n-1}(t) - E_{2n}(t) + E_{2n}(0) - (1-t)E_{2n-1}(t) + E_{2n}(1) - E_{2n}(t)] \\ = (-1)^n [2E_{2n}(t) + (1-2t)E_{2n-1}(t)].$$

Next, since the derivative of the right side of (28) is $-(1-2t)(-1)^{n-1}E_{2n-2}(t)$ and $(-1)^{n-1}E_{2n-2}(t) \geq 0$ for all $t \in [0, 1]$, it is immediate that

$$\int_0^1 |G'_n(t, s)| \, ds \leq (-1)^n \max\{E_{2n-1}(0), -E_{2n-1}(1)\}.$$

However, since $E_{2n-1}(0) + E_{2n-1}(1) = 0$ (Jordan [16, p. 291]) it follows that

$$\int_0^1 |G'_n(t, s)| \, ds \leq (-1)^n E_{2n-1}(0) = (-1)^n E'_{2n}(0) = (-1)^{n+1} \frac{2(2^{2n}-1)}{(2n)!} B_{2n},$$

where in the last equality we have used (18) and (21).

Remark 2.9

As in Remark 2.8, we have

$$x^{(2i+1)}(t) = \sum_{k=0}^{m-i-1} [x^{(2k+2i)}(0)A'_k(1-t) + x^{(2k+2i)}(1)A'_k(t)] + \int_0^1 G'_{m-i}(t, s)x^{(2m)}(s) \, ds$$

and hence Lemma 2.4 gives the best possible inequalities

$$\begin{aligned} & \left| x^{(2i+1)}(t) - \sum_{k=0}^{m-i-1} [x^{(2k+2i)}(0)A_k'(1-t) + x^{(2k+2i)}(1)A_k'(t)] \right| \\ & \leq (-1)^{m-i} [2E_{2m-2i}(t) + (1-2t)E_{2m-2i-1}(t)]M_{2m} \\ & \leq (-1)^{m-i+1} \frac{2(2^{2m-2i}-1)}{(2m-2i)!} B_{2m-2i} M_{2m}; \quad i = 0, 1, \dots, m-1. \end{aligned} \quad (31)$$

A similar result is also obtained by Varma and Howell [20].

Lemma 2.5

The following equality holds:

$$\int_0^1 |G_n(t, s)| \sin \pi s \, ds = \frac{1}{\pi^{2n}} \sin \pi t. \quad (32)$$

Proof. Since

$$\begin{aligned} \int_0^1 |G_1(t, s)| \sin \pi s \, ds &= (1-t) \int_0^t s \sin \pi s \, ds + \int_t^1 (1-s) \sin \pi s \, ds \\ &= (1-t) \left(-\frac{1}{\pi} t \cos \pi t + \frac{1}{\pi^2} \sin \pi t \right) + t \left((1-t) \frac{1}{\pi} \cos \pi t + \frac{1}{\pi^2} \sin \pi t \right) \\ &= \frac{1}{\pi^2} \sin \pi t \end{aligned}$$

the equality (32) easily follows from (5) by using an inductive argument.

Lemma 2.6

The following inequality holds:

$$\frac{1}{\pi^{2n+1}} \sin \pi t \leq (-1)^n \Lambda_n(t), \quad 0 \leq t \leq 1. \quad (33)$$

Proof. Since $\sin \pi t \leq \pi t$, $0 \leq t \leq 1$, it follows that

$$(-1)^n \Lambda_n(t) = \int_0^1 |G_n(t, s)| s \, ds \geq \int_0^1 |G_n(t, s)| \frac{\sin \pi s}{\pi} \, ds$$

and now (33) follows from (32).

Remark 2.10

Widder [7] has proved that

$$\frac{t^{n+1}(1-t)^{n+1}}{(n+2)!} \leq (-1)^n \Lambda_n(t), \quad 0 \leq t \leq 1. \quad (34)$$

We note that (33) is sharper than (34). In fact, from the elementary inequality

$$t(1-t) \leq \frac{1}{\pi} \sin \pi t, \quad 0 \leq t \leq 1, \quad (35)$$

we find that

$$\frac{t^{n+1}(1-t)^{n+1}}{(n+2)!} \leq \frac{1}{\pi^{n+1}} \frac{1}{(n+2)!} (\sin \pi t)^{n+1} \leq \frac{1}{\pi^{2n+1}} \sin \pi t.$$

Remark 2.11

From (9) and (33) it follows that

$$(-1)^n B_{2n+1} \left(\frac{1+t}{2} \right) \geq \frac{(2n+1)!}{(2\pi)^{2n+1}} \sin \pi t, \quad 0 \leq t \leq 1. \quad (36)$$

Lemma 2.7

The following inequality holds:

$$(-1)^n A_n(t) \leq \frac{1}{\pi^{2n}} \left(\frac{\pi}{3} \right) \sin \pi t, \quad 0 \leq t \leq 1. \quad (37)$$

Proof. Since from (10)

$$(-1)A_1(t) = \frac{1}{6}t(1-t)(1+t) \leq \frac{1}{3}t(1-t)$$

inequality (35) gives

$$(-1)A_1(t) \leq \frac{1}{3\pi} \sin \pi t$$

and hence (36) is true for $n = 1$.

Next, since from (6)

$$(-1)^{m+1} A_{m+1}(t) = \int_0^1 |G_1(t, s)| A_m(s) \, ds$$

if (37) is true for $m \geq 1$, then from (32) for $n = 1$ it follows that

$$(-1)^{m+1} A_{m+1}(t) \leq \int_0^1 |G_1(t, s)| \frac{1}{\pi^{2m}} \left(\frac{\pi}{3} \right) \sin \pi s \, ds = \frac{1}{\pi^{2m+2}} \left(\frac{\pi}{3} \right) \sin \pi t.$$

Remark 2.12

Widder [8] has proved that there exists a constant M such that $(-1)^n A_n(t) \leq M/\pi^{2n}$. From (37) it is clear that $M \leq \pi/3$. This can also be proved directly from (7) as follows:

$$(-1)^n A_n(t) \leq \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+1}} \leq \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{\pi^{2n}} \left(\frac{\pi}{3} \right). \quad (38)$$

Further, from (7) we also have

$$|A'_n(t)| \leq \frac{2}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \leq \frac{2}{\pi^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{\pi^{2n}} \left(\frac{\pi^2}{3} \right). \quad (39)$$

Remark 2.13

From (9) and (37) it is immediate that

$$(-1)^n B_{2n+1} \left(\frac{1+t}{2} \right) \leq \frac{(2n+1)!}{(2\pi)^{2n}} \left(\frac{\pi}{6} \right) \sin \pi t, \quad 0 \leq t \leq 1.$$

Remark 2.14

From (33) and (37) it is obvious that

$$\frac{1}{\pi^{2n+1}} \sin \pi t \leq (-1)^n A_n(1-t) \leq \frac{1}{\pi^{2n}} \left(\frac{\pi}{3} \right) \sin \pi t, \quad 0 \leq t \leq 1.$$

Lemma 2.8

The following inequality holds:

$$\int_0^1 |G_n(t, s)| \, ds \leq \frac{1}{\pi^{2n-2}} \left(\frac{1}{2\pi} \right) \sin \pi t, \quad 0 \leq t \leq 1. \quad (40)$$

Proof. Since from (21)

$$\int_0^1 |G_1(t, s)| ds = \frac{1}{2} t(1-t),$$

inequality (35) gives

$$\int_0^1 |G_1(t, s)| ds \leq \frac{1}{2\pi} \sin \pi t$$

and hence (40) is true for $n = 1$.

Next, if (40) is true for $m \geq 1$, then from (5) we have

$$\begin{aligned} \int_0^1 |G_{m+1}(t, s)| ds &\leq (1-t) \int_0^1 \int_0^t |G_m(t_1, s)| t_1 dt_1 ds + t \int_0^1 \int_t^1 |G_m(t_1, s)| (1-t_1) dt_1 ds \\ &\leq \left[(1-t) \int_0^t t_1 \sin \pi t_1 dt_1 + t \int_t^1 (1-t_1) \sin \pi t_1 dt_1 \right] \frac{1}{\pi^{2m-2}} \left(\frac{1}{2\pi} \right) \\ &= \frac{1}{\pi^{2m}} \left(\frac{1}{2\pi} \right) \sin \pi t. \end{aligned}$$

Remark 2.15

From (26) and (40) it follows that

$$|E_{2n}(t)| \leq \frac{1}{\pi^{2n-2}} \left(\frac{1}{2\pi} \right) \sin \pi t \leq \frac{1}{2\pi^{2n-1}}. \quad (41)$$

Jordan [16, p. 302] has proved that $|E_{2n}(t)| < 2/(3\pi^{2n-1})$, and hence (41) gives a better estimate. Further, from (41) we find that

$$|E_{2n}(\tfrac{1}{2})| \leq \frac{1}{2\pi^{2n-1}}$$

and hence from the formula $E_{2n} = 2^{2n}(2n)!E_{2n}(\tfrac{1}{2})$ it is immediate to have

$$|E_{2n}| \leq \left(\frac{2}{\pi} \right)^{2n-1} (2n)! \quad (42)$$

which is sharper than

$$|E_{2n}| < \frac{4}{3} \left(\frac{2}{\pi} \right)^{2n-1} (2n)!$$

given by Jordan [16, p. 303].

Lemma 2.9

If $(-1)^n x^{(2n)}(t)$ is nonnegative and concave in $[0, 1]$ then

$$\left| \int_0^1 G_n(t, s) x^{(2n)}(s) ds \right| \geq \frac{2M_{2n}}{\pi^{2n+2}} \sin \pi t, \quad 0 \leq t \leq 1, \quad (43)$$

where

$$M_{2n} = \max_{0 \leq t \leq 1} (-1)^n x^{(2n)}(t).$$

Proof. Since $(-1)^n x^{(2n)}(t)$ is concave in $[0, 1]$, following Boas [3] we have $(-1)^n x^{(2n)}(t) \geq M_{2n}t(1-t)$. Thus, from the elementary inequality

$$t(1-t) \geq \frac{2}{\pi^2} \sin \pi t \quad (44)$$

it follows that $(-1)^n x^{(2n)}(t) \geq (2/\pi^2) M_{2n} \sin \pi t$.

Now using the fact that $(-1)^n x^{(2n)}(t) \geq 0$, and the equality (32) to obtain

$$\begin{aligned} \left| \int_0^1 G_n(t, s) x^{(2n)}(s) ds \right| &= \left| \int_0^1 |G_n(t, s)| (-1)^n x^{(2n)}(s) ds \right| \\ &\geq \frac{2}{\pi^2} M_{2n} \int_0^1 |G_n(t, s)| \sin \pi s ds \\ &= \frac{2M_{2n}}{\pi^{2n+2}} \sin \pi t. \end{aligned}$$

Remark 2.16

In particular if $t = \frac{1}{2}$, inequality (43) reduces to

$$\left| \int_0^1 G_n\left(\frac{1}{2}, s\right) x^{(2n)}(s) ds \right| \geq \frac{2M_{2n}}{\pi^{2n+2}}$$

which is sharper than

$$\left| \int_0^1 G_n\left(\frac{1}{2}, s\right) x^{(2n)}(s) ds \right| \geq \frac{4M_{2n}}{\pi^{2n+3}} (1 - 3^{-2n-1})$$

obtained by Boas [3].

Lemma 2.10

The following inequality holds:

$$\int_0^1 |G'_n(t, s)| ds \leq \frac{1}{\pi^{2n-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1-2t) \cos \pi t], \quad 0 \leq t \leq 1. \quad (45)$$

Proof. From the inequality (40) for $n > 1$ it is immediate to obtain

$$\begin{aligned} \int_0^1 |G'_n(t, s)| ds &\leq \frac{1}{\pi^{2n-4}} \left(\frac{1}{2\pi} \right) \left(\int_0^t t_1 \sin \pi t_1 dt_1 + \int_t^1 (1-t_1) \sin \pi t_1 dt_1 \right) \\ &= \frac{1}{\pi^{2n-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \end{aligned}$$

Further, for $n = 1$ we have

$$\int_0^1 |G'_n(t, s)| ds = \frac{t^2 + (1-t)^2}{2},$$

however, since

$$t^2 + (1-t)^2 \leq \frac{1}{\pi} [2 \sin \pi t + \pi(1-2t) \cos \pi t], \quad 0 \leq t \leq 1,$$

the inequality (45) is true for $n = 1$ also.

Lemma 2.11

The following inequality holds:

$$\int_0^1 |G'_n(t, s)| \sin \pi s ds \leq \frac{1}{\pi^{2n}} [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \quad (46)$$

Proof. For $n = 1$, the direct computation gives

$$\begin{aligned} \int_0^1 |G'_n(t, s)| \sin \pi s ds &= \int_0^t s \sin \pi s ds + \int_t^1 (1-s) \sin \pi s ds \\ &= \frac{1}{\pi^2} [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \end{aligned}$$

For $n > 1$, as earlier we have

$$\begin{aligned} \int_0^1 |G'_n(t, s)| \sin \pi s \, ds &\leq \int_0^t t_1 \left(\int_0^1 |G_{n-1}(t_1, s)| \sin \pi s \, ds \right) dt_1 \\ &\quad + \int_t^1 (1-t_1) \left(\int_0^1 |G_{n-1}(t_1, s)| \sin \pi s \, ds \right) dt_1 \\ &= \frac{1}{\pi^{2n-2}} \left(\int_0^t t_1 \sin \pi t_1 \, dt_1 + \int_t^1 (1-t_1) \sin \pi t_1 \, dt_1 \right) \\ &= \frac{1}{\pi^{2n}} [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \end{aligned}$$

Lemma 2.12

The following inequality holds:

$$\int_0^1 |G_n(t, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \leq \frac{1}{\pi^{2n-2}} \left(\frac{4}{\pi^2} \right) \sin \pi t. \quad (47)$$

Proof. For $n = 1$, the direct computation gives (47). If (47) is true for $n = m \geq 1$, then as earlier we have

$$\begin{aligned} \int_0^1 |G_{m+1}(t, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \\ \leq (1-t) \int_0^t t_1 \left(\int_0^1 |G_m(t_1, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \right) dt_1 \\ + t \int_t^1 (1-t_1) \left(\int_0^1 |G_m(t_1, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \right) dt_1 \\ \leq \frac{1}{\pi^{2m-2}} \left(\frac{4}{\pi^2} \right) \left((1-t) \int_0^t t_1 \sin \pi t_1 \, dt_1 + t \int_t^1 (1-t_1) \sin \pi t_1 \, dt_1 \right) \\ = \frac{1}{\pi^{2m}} \left(\frac{4}{\pi^2} \right) \sin \pi t. \end{aligned}$$

Lemma 2.13

The following inequality holds:

$$\int_0^1 |G'_n(t, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \leq \frac{1}{\pi^{2n-2}} \left(\frac{4}{\pi^2} \right) [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \quad (48)$$

Proof. For $n = 1$, the proof is by direct computation. If (48) is true for $n = m \geq 1$, then as an application of (47) we find

$$\begin{aligned} \int_0^1 |G'_{m+1}(t, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \\ \leq \int_0^t t_1 \left(\int_0^1 |G_m(t_1, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \right) dt_1 \\ + \int_t^1 (1-t_1) \left(\int_0^1 |G_m(t_1, s)| [2 \sin \pi s + \pi(1-2s) \cos \pi s] \, ds \right) dt_1 \\ \leq \frac{1}{\pi^{2m-2}} \left(\frac{4}{\pi^2} \right) \left(\int_0^t t_1 \sin \pi t_1 \, dt_1 + \int_t^1 (1-t_1) \sin \pi t_1 \, dt_1 \right) \\ = \frac{1}{\pi^{2m}} \left(\frac{4}{\pi^2} \right) [2 \sin \pi t + \pi(1-2t) \cos \pi t]. \end{aligned}$$

Lemma 2.14 [10]

Let B be a Banach space and let $0 < r \in R$, $\bar{S}(x_0, r) = \{x \in B: \|x - x_0\| \leq r\}$. Let T map $\bar{S}(x_0, r)$ into B and

- (i) for all $x, y \in \bar{S}(x_0, r)$, $\|Tx - Ty\| \leq \alpha \|x - y\|$, where $0 < \alpha < 1$,
- (ii) $r_0 = (1 - \alpha)^{-1} \|Tx_0 - x_0\| \leq r$.

Then, the following hold:

- (1) T has a fixed point x^* in $\bar{S}(x_0, r_0)$;
- (2) x^* is the unique fixed point of T in $\bar{S}(x_0, r_0)$;
- (3) the sequence $\{x_n\}$, where $x_{n+1} = Tx_n$; $n = 0, 1, \dots$ converges to x^* with

$$\|x^* - x_n\| \leq \alpha^n r_0, \quad \text{and} \quad \|x^* - x_n\| \leq \alpha(1 - \alpha)^{-1} \|x_n - x_{n-1}\|;$$

- (4) for any $x \in \bar{S}(x_0, r_0)$,

$$x^* = \lim_{n \rightarrow \infty} T^n x.$$

Lemma 2.15 [10]

Let (E, \leq) be a partially ordered space and $x_0 \leq y_0$ be two elements of E , and $[x_0, y_0]$ denotes the interval $\{x \in E: x_0 \leq x \leq y_0\}$. Further, let $T: [x_0, y_0] \rightarrow E$ be an isotone operator $[T(x) \leq T(y)]$, whenever $x \leq y$ and let it possess the properties

- (i) $x_0 \leq T(x_0)$;
- (ii) the (nondecreasing) sequence $\{T^n(x_0)\}$ where $T^0(x_0) = x_0$, $T^{n+1}(x_0) = T[T^n(x_0)]$ for each $n = 0, 1, \dots$ is well defined, i.e. $T^n(x_0) \leq y_0$ for each natural n ;
- (iii) the sequence $\{T^n(x_0)\}$ has sup $x \in E$, i.e. $T^n(x_0) \uparrow x$;
- (iv) $T^{n+1}(x_0) \uparrow T(x)$.
- (i)' $T(y_0) \leq y_0$;
- (ii)' the (nonincreasing) sequence $\{T^n(y_0)\}$ is well defined, i.e. $T^n(y_0) \geq x_0$ for each natural n ;
- (iii)' the sequence $\{T^n(y_0)\}$ has inf $y \in E$, i.e., $T^n(y_0) \downarrow y$;
- (iv)' $T^{n+1}(y_0) \downarrow T(y)$.

Then, $x = T(x)$ and for any other fixed point $z \in [x_0, y_0]$ of T , $x \leq z$ is true. [Then, $y = T(y)$ and for any other fixed point $z \in [x_0, y_0]$ of T , $z \leq y$ is valid.]

Moreover, if T possesses both properties (i) and (i)', then the sequences $\{T^n(x_0)\}$, $\{T^n(y_0)\}$ are well defined and if, further, T has the properties (iii), (iii)' and (iv), (iv)' then

$$x_0 \leq T(x_0) \leq \dots \leq T^n(x_0) \leq \dots \leq x \leq y \leq \dots \leq T^n(y_0) \leq \dots \leq T(y_0) \leq y_0$$

and $x = T(x)$, $y = T(y)$ also any other fixed point $z \in [x_0, y_0]$ of T satisfies $x \leq z \leq y$.

3. EXISTENCE AND UNIQUENESS

Theorem 3.1

Suppose that:

- (i) K_i , $0 \leq i \leq q$ are given real numbers and let Q be the maximum of $|f(t, x_0, x_1, \dots, x_q)|$ on the compact set: $[0, 1] \times D_0$, where $D_0 = \{(x_0, x_1, \dots, x_q): |x_i| \leq 2K_i, 0 \leq i \leq q\}$;
- (ii)

$$Q \frac{(-1)^{m-i} E_{2m-2i}}{2^{2m-2i} (2m-2i)!} \leq K_{2i}; \quad i = 0, 1, \dots, \left[\frac{q}{2} \right];$$

- (iii)

$$Q \frac{(-1)^{m-i+1} 2(2^{2m-2i} - 1) B_{2m-2i}}{(2m-2i)!} \leq K_{2i+1}; \quad i = 0, 1, \dots, \left[\frac{q-1}{2} \right];$$

- (iv)

$$\max\{|\alpha_i|, |\beta_i|\} + \left(\sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \frac{(-1)^k E_{2k}}{2^{2k} (2k)!} \right) = C_{2i} \leq K_{2i}; \quad i = 0, 1, \dots, \left[\frac{q}{2} \right];$$

(v)

$$|\alpha_i - \beta_i| + \left(\sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \frac{(-1)^{k+1} 2(2^k - 1) B_{2k}}{(2k)!} \right) = C_{2i+1} \leq K_{2i+1};$$

$$i = 0, 1, \dots, \left[\frac{q-1}{2} \right].$$

Then, the boundary value problem (1), (2) has a solution in D_0 .

Proof. From Lemma 2.1 it follows that the boundary value problem (1), (2) is equivalent to the following Fredholm type of integral equation:

$$x(t) = \sum_{k=0}^{m-1} [\alpha_k A_k(1-t) + \beta_k A_k(t)] + \int_0^1 |G_m(t, s)| f(s, x(s), x'(s), \dots, x^{(q)}(s)) ds. \quad (49)$$

Next, we define the set

$$B[0, 1] = \{x(t) \in C^{(q)}[0, 1]: \|x^{(i)}\| = \max_{0 \leq t \leq 1} |x^{(i)}(t)| \leq 2K_i, 0 \leq i \leq q\}.$$

It is easy to verify that $B[0, 1]$ is a closed convex subset of the Banach space $C^{(q)}[0, 1]$. Consider an operator $T: C^{(q)}[0, 1] \rightarrow C^{(2m)}[0, 1]$ as follows:

$$(Tx)(t) = \sum_{k=0}^{m-1} [\alpha_k A_k(1-t) + \beta_k A_k(t)] + \int_0^1 |G_m(t, s)| f(s, x(s), x'(s), \dots, x^{(q)}(s)) ds. \quad (50)$$

Obviously, any fixed point of (50) is a solution of the boundary value problem (1), (2).

We shall show that T maps $B[0, 1]$ into itself. For this, let $x(t) \in B[0, 1]$ then from (50), (14), (26) and hypotheses (i), (ii), (iv) we find

$$\begin{aligned} |(Tx)^{(2i)}(t)| &\leq \sum_{k=0}^{m-i-1} [|\alpha_{k+i}| A_k(1-t) + |\beta_{k+i}| A_k(t)] + Q \int_0^1 |G_{m-i}(t, s)| ds \\ &\leq \max_{0 \leq t \leq 1} [|\alpha_i|(1-t) + |\beta_i|t] + \sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \\ &\quad \times \int_0^1 |G_k(t, s)|(s+1-s) ds + Q \int_0^1 |G_{m-i}(t, s)| ds \\ &\leq \max\{|\alpha_i|, |\beta_i|\} + \sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \frac{(-1)^k E_{2k}}{2^{2k} (2k)!} + Q \frac{(-1)^{m-i} E_{2m-2i}}{2^{2m-2i} (2m-2i)!} \\ &\leq K_{2i} + K_{2i} \\ &= 2K_{2i}; \quad i = 0, 1, \dots, \left[\frac{q}{2} \right]. \end{aligned} \quad (51)$$

Similarly, from (50), (14), (29) and hypotheses (i), (iii), (v) we get

$$\begin{aligned} |(Tx)^{(2i+1)}(t)| &\leq |\alpha_i A'_0(1-t) + \beta_i A'_0(t)| + \sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \\ &\quad \times [|\Lambda'_k(t)| + |\Lambda'_k(1-t)|] + Q \int_0^1 |G'_{m-i}(t, s)| ds \\ &\leq |\alpha_i - \beta_i| + \sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \int_0^1 |G'_k(t, s)|(s+1-s) ds \\ &\quad + Q \int_0^1 |G'_{m-i}(t, s)| ds \end{aligned}$$

$$\begin{aligned}
&\leq |\alpha_i - \beta_i| + \sum_{k=1}^{m-i-1} \max\{|\alpha_{k+i}|, |\beta_{k+i}|\} \frac{(-1)^{k+1} 2(2^{2k} - 1) B_{2k}}{(2k)!} \\
&\quad + Q \frac{(-1)^{m-i+1} 2(2^{2m-2i} - 1) B_{2m-2i}}{(2m-2i)!} \\
&\leq K_{2i+1} + K_{2i+1} \\
&= 2K_{2i+1}; \quad i = 0, 1, \dots, \left\lfloor \frac{q-1}{2} \right\rfloor.
\end{aligned} \tag{52}$$

This completes the proof of $TB[0, 1] \subseteq B[0, 1]$. The inequalities (51) and (52) imply that the sets $\{(Tx)^{(i)}(t): x(t) \in B[0, 1]\}$, $0 \leq i \leq q$ are uniformly bounded and equicontinuous on $[0, 1]$. Hence $TB[0, 1]$ is compact follows from the Ascoli–Arzela theorem. The Schauder fixed point theorem is applicable and a fixed point of T in D_0 exists.

Corollary 3.2

Assume that the function $f(t, x_0, x_1, \dots, x_q)$ on $[0, 1] \times R^{q+1}$ satisfies the following condition:

$$|f(t, x_0, x_1, \dots, x_q)| \leq L + \sum_{i=0}^q L_i |x_i|^{\alpha_i}, \tag{53}$$

where L, L_i , $0 \leq i \leq q$ are nonnegative constants, and $0 \leq \alpha_i < 1$, $0 \leq i \leq q$. Then, the boundary value problem (1), (2) has a solution.

Proof. For $x(t) \in B[a, b]$ the condition (53) implies that

$$|f(t, x(t), x'(t), \dots, x^{(q)}(t))| \leq L + \sum_{i=0}^q L_i (2K_i)^{\alpha_i} = Q_1, \quad \text{say.}$$

Now Corollary 3.2 follows immediately by observing that hypotheses of Theorem 3.1 are satisfied with Q replaced by Q_1 provided K_i , $0 \leq i \leq q$ are sufficiently large.

Theorem 3.3

Suppose that the function $f(t, x_0, x_1, \dots, x_q)$ on $[0, 1] \times D_1$ satisfies the following condition:

$$|f(t, x_0, x_1, \dots, x_q)| \leq L + \sum_{i=0}^q L_i |x_i|, \tag{54}$$

where

$$\begin{aligned}
D_1 = &\left\{ (x_0, x_1, \dots, x_q): |x_{2i}| \leq (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C \sin \pi t + C_{2i}, \quad 0 \leq i \leq \left\lfloor \frac{q}{2} \right\rfloor, \right. \\
&\left. |x_{2i+1}| \leq (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C [2 \sin \pi t + \pi(1-2t) \cos \pi t] + C_{2i+1}, \quad 0 \leq i \leq \left\lfloor \frac{q-1}{2} \right\rfloor \right\}
\end{aligned}$$

and

$$C = L + \sum_{i=0}^{\lfloor q/2 \rfloor} L_{2i} C_{2i} + \sum_{i=0}^{\lfloor (q-1)/2 \rfloor} L_{2i+1} C_{2i+1}, \tag{55}$$

$$\theta = \sum_{i=0}^{\lfloor q/2 \rfloor} L_{2i} \frac{1}{\pi^{2m-2i}} + \sum_{i=0}^{\lfloor (q-1)/2 \rfloor} L_{2i+1} \frac{1}{\pi^{2m-2i-2}} \left(\frac{4}{\pi^2} \right) < 1. \tag{56}$$

Then, the boundary value problem (1), (2) has a solution in D_1 .

Proof. Let

$$y(t) = x(t) - \sum_{k=0}^{m-1} [\alpha_k \Lambda_k(1-t) + \beta_k \Lambda_k(t)],$$

so that the boundary value problem (1), (2) is equivalent to the following:

$$(-1)^m y^{(2m)}(t) = f(t, y(t) + P_{2m-1}(t), \quad y'(t) + P'_{2m-1}(t), \dots, y^{(q)}(t) + P_{2m-1}^{(q)}(t)), \tag{57}$$

$$y^{(2i)}(0) = y^{(2i)}(1) = 0; \quad i = 0, 1, \dots, m-1, \quad (58)$$

where

$$P_{2m-1}(t) = \sum_{k=0}^{m-1} [\alpha_k \Lambda_k(1-t) + \beta_k \Lambda_k(t)].$$

Define $M[0, 1]$ as the space of q times continuously differentiable functions. If we introduce in $M[0, 1]$ the finite norm

$$\|y\| = \max \left\{ \sup_{0 \leq i \leq 1} \frac{|y^{(2i)}(t)|}{\sin \pi t}, \quad 0 \leq i \leq \left[\frac{q}{2} \right]; \quad \frac{|y^{(2i+1)}(t)|}{2 \sin \pi t + \pi(1-2t)\cos \pi t}, \quad 0 \leq i \leq \left[\frac{q-1}{2} \right] \right\}$$

then it becomes a Banach space. As in Theorem 3.1, we shall show that $T: M[0, 1] \rightarrow M[0, 1]$ defined by

$$(Ty)(t) = \int_0^1 |G_m(t, s)| f(s, y(s) + P_{2m-1}(s), \dots, y^{(q)}(s) + P_{2m-1}^{(q)}(s)) \, ds \quad (59)$$

maps the set

$$B_1[0, 1] = \left\{ y(t) \in M[0, 1]: \|y\| \leq (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C \right\}$$

into itself. For this, if $y(t) \in B_1[0, 1]$ then it immediately follows that

$$|y^{(2i)}(t)| \leq (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right],$$

$$|y^{(2i+1)}(t)| \leq (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C [2 \sin \pi t + \pi(1-2t)\cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right].$$

Hence, $(y(t) + P_{2m-1}(t), y'(t) + P_{2m-1}'(t), \dots, y^{(q)}(t) + P_{2m-1}^{(q)}(t)) \in D_1$. Thus, from (59), (54), (40), (32) and (47) it follows that

$$\begin{aligned} |(Ty)^{(2i)}(t)| &\leq \int_0^1 |G_{m-i}(t, s)| \left(L + \sum_{i=0}^q L_i |y^{(i)}(s) + P_{2m-1}^{(i)}(s)| \right) \, ds \\ &\leq \int_0^1 |G_{m-i}(t, s)| \left(L + \sum_{i=0}^{[q/2]} L_{2i} |P_{2m-1}^{(2i)}(s)| + \sum_{i=0}^{[(q-1)/2]} L_{2i+1} |P_{2m-1}^{(2i+1)}(s)| \right. \\ &\quad + \sum_{i=0}^{[q/2]} L_{2i} \sin \pi s \frac{|y^{(2i)}(s)|}{\sin \pi s} \\ &\quad + \sum_{i=0}^{[(q-1)/2]} L_{2i+1} [2 \sin \pi s + \pi(1-2s)\cos \pi s] \frac{|y^{(2i+1)}(s)|}{2 \sin \pi s + \pi(1-2s)\cos \pi s} \Big) \, ds \\ &\leq \int_0^1 |G_{m-i}(t, s)| \left(L + \sum_{i=0}^{[q/2]} L_{2i} C_{2i} + \sum_{i=0}^{[(q-1)/2]} L_{2i+1} C_{2i+1} + \sum_{i=0}^{[q/2]} L_{2i} \sin \pi s \|y\| \right. \\ &\quad + \sum_{i=0}^{[(q-1)/2]} L_{2i+1} [2 \sin \pi s + \pi(1-2s)\cos \pi s] \|y\| \Big) \, ds \\ &\leq C \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t + \sum_{i=0}^{[q/2]} L_{2i} \frac{1}{\pi^{2m-2i}} \sin \pi t \|y\| \\ &\quad + \sum_{i=0}^{[(q-1)/2]} L_{2i+1} \frac{1}{\pi^{2m-2i-2}} \left(\frac{4}{\pi^2} \right) \sin \pi t \|y\| \\ &\leq \left[C \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) + \theta \|y\| \right] \sin \pi t, \quad 0 \leq i \leq [q/2]. \end{aligned} \quad (60)$$

Similarly, from (59), (54), (45), (46) and (48) we have

$$|(Ty)^{(2i+1)}(t)| \leq \left[C \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) + \theta \|y\| \right] [2 \sin \pi s + \pi(1-2s) \cos \pi s], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right]. \quad (61)$$

Now combining (60) and (61), to obtain

$$\begin{aligned} \|(Ty)\| &\leq C \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) + \theta(1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C \\ &= (1-\theta)^{-1} \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) C. \end{aligned}$$

Remark 3.1

In Theorem 3.3 the inequality (56) for $q = 0$ is the best possible, i.e. θ cannot be replaced by a smaller number. Indeed, in case of equality $L_0 = \pi^{2m}$ the boundary value problem $(-1)^m x^{(2m)} = L_0 x$; $x(0) = \epsilon (\neq 0)$, $x^{(2i)}(0) = 0$, $1 \leq i \leq m-1$; $x^{(2i)}(1) = 0$, $0 \leq i \leq m-1$ has no solution.

Theorem 3.4

Suppose that the differential equation (1) together with the boundary conditions

$$x^{(2i)}(0) = x^{(2i)}(1) = 0, \quad 0 \leq i \leq m-1 \quad (62)$$

has a nontrivial solution $x(t)$ and the condition (54) with $L = 0$ on $[0, 1] \times D_2$ is satisfied, where

$$\begin{aligned} D_2 = \left\{ (x_0, x_1, \dots, x_q) : |x_{2i}| \leq (-1)^{m-i} E_{2m-2i}(t) \max_{0 \leq t \leq 1} |x^{(2m)}(t)|, \quad 0 \leq i \leq \left[\frac{q}{2} \right], \right. \\ \left. |x_{2i+1}| \leq (-1)^{m-i} [2E_{2m-2i}(t) + (1-2t)E_{2m-2i-1}(t)] \max_{0 \leq t \leq 1} |x^{(2m)}(t)|, \quad 0 \leq i \leq \left[\frac{q-1}{2} \right] \right\}. \end{aligned}$$

Then, it is necessary that $\theta \geq 1$.

Proof. From (27) and (31) for any function $x(t) \in C^{(2m)}[0, 1]$ satisfying (62) it follows that $(x(t), x'(t), \dots, x^{(q)}(t)) \in D_2$. Now since $x(t)$ is a nontrivial solution of (1), (62) we find that

$$\eta_i = \sup_{0 \leq t \leq 1} \frac{|x^{(2i)}(t)|}{\sin \pi t}, \quad 0 \leq i \leq \left[\frac{q}{2} \right]; \quad v_i = \sup_{0 \leq t \leq 1} \frac{|x^{(2i+1)}(t)|}{2 \sin \pi t + \pi(1-2t) \cos \pi t}, \quad 0 \leq i \leq \left[\frac{q-1}{2} \right]$$

exist and must be different from zero. Now as in Theorem 3.3 it is easy to obtain

$$\eta_i \leq \theta \max \left\{ \eta_i, 0 \leq i \leq \left[\frac{q}{2} \right]; v_i, 0 \leq i \leq \left[\frac{q-1}{2} \right] \right\}, \quad 0 \leq i \leq \left[\frac{q}{2} \right]$$

and

$$v_i \leq \theta \max \left\{ \eta_i, 0 \leq i \leq \left[\frac{q}{2} \right]; v_i, 0 \leq i \leq \left[\frac{q-1}{2} \right] \right\}, \quad 0 \leq i \leq \left[\frac{q-1}{2} \right].$$

Hence, it is necessary that $\theta \geq 1$.

Remark 3.2

In Theorem 3.4 the inequality $\theta \geq 1$ for $q = 0$ is the best possible. Indeed, in case of equality $L_0 = \pi^{2m}$ the boundary value problem $(-1)^m x^{(2m)} = L_0 x$, (62) has nontrivial solutions $x(t) = c \sin \pi t$, where c is an arbitrary constant.

Remark 3.3

If the condition (54) with $L = 0$ is satisfied, then obviously $x(t) \equiv 0$ is a solution of (1), (62); if $\theta < 1$ then Theorem 3.4 also guarantees its uniqueness in D_2 .

Theorem 3.5

Suppose that the function $f(t, x_0, x_1, \dots, x_q)$ on $[0, 1] \times D'_1$ satisfies the Lipschitz condition

$$|f(t, x_0, x_1, \dots, x_q) - f(t, \bar{x}_0, \bar{x}_1, \dots, \bar{x}_q)| \leq \sum_{i=0}^q L_i |x_i - \bar{x}_i|, \quad (63)$$

where D'_1 is same as D_1 with

$$L = \max_{0 \leq i \leq 1} |f(t, 0, 0, \dots, 0)|.$$

Then, the boundary value problem (1), (2) has a unique solution in D'_1 .

Proof. Since the Lipschitz condition (63) implies (54), the existence of a solution follows from Theorem 3.3. To prove the uniqueness let $x(t)$ and $y(t)$ be two solutions of the boundary value problem (1), (2) in D'_1 . Then, as in Theorem 3.3 it follows that $\|x - y\| \leq \theta \|x - y\|$, and since $\theta < 1$ it follows that $\|x - y\| = 0$, i.e. $x(t) \equiv y(t)$.

Remark 3.4

Once again in Theorem 3.5 the inequality $\theta < 1$ for $q = 0$ is the best possible. Indeed in case of equality the problem $(-1)^m x^{(2m)} = L_0 x$, (62) has infinite number of solutions.

4. PICARD'S ITERATIONS**Definition 4.1**

A function $\bar{x}(t) \in C^{(2m)}[0, 1]$ is called an approximate solution of the boundary value problem (1), (2) if there exist δ and ϵ , nonnegative constants, such that

$$|(-1)^m \bar{x}^{(2m)}(t) - f(t, \bar{x}(t), \bar{x}'(t), \dots, \bar{x}^{(q)}(t))| \leq \delta$$

and

$$|P_{2m-1}^{(2i)}(t) - \bar{P}_{2m-1}^{(2i)}(t)| \leq \epsilon \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right],$$

$$|P_{2m-1}^{(2i+1)}(t) - \bar{P}_{2m-1}^{(2i+1)}(t)| \leq \epsilon \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1-2t)\cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right],$$

where

$$\bar{P}_{2m-1}(t) = \sum_{k=0}^{m-1} [\bar{x}^{(2k)}(0) \Lambda_k(1-t) + \bar{x}^{(2k)}(1) \Lambda_k(t)].$$

The approximate solution $\bar{x}(t)$ can be expressed as

$$\bar{x}(t) = \bar{P}_{2m-1}(t) + \int_0^1 |G_m(t, s)| [f(s, \bar{x}(s), \bar{x}'(s), \dots, \bar{x}^{(q)}(s)) + \eta(s)] ds, \quad (64)$$

where

$$\eta(t) = (-1)^m \bar{x}^{(2m)}(t) - f(t, \bar{x}(t), \bar{x}'(t), \dots, \bar{x}^{(q)}(t)) \quad \text{and} \quad \max_{0 \leq t \leq 1} |\eta(t)| \leq \delta.$$

Theorem 4.1

With respect to the boundary value problem (1), (2) we assume that there exists an approximate solution $\bar{x}(t)$ and

(i) the function $f(t, x_0, x_1, \dots, x_q)$ satisfies the Lipschitz condition (63) on $[0, 1] \times D_3$, where

$$D_3 = \left\{ (x_0, x_1, \dots, x_q) : |x_{2i} - \bar{x}^{(2i)}(t)| \leq N \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right], \right. \\ \left. |x_{2i+1} - \bar{x}^{(2i+1)}(t)| \leq N [2 \sin \pi t + \pi(1-2t)\cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right] \right\};$$

(ii)

$$N_0 = (1 - \theta)^{-1}(\epsilon + \delta) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \leq N.$$

Then, the following hold

- (1) there exists a solution $x^*(t)$ of (1), (2) in $\bar{S}(\bar{x}, N_0)$,
- (2) $x^*(t)$ is the unique solution of (1), (2) in $\bar{S}(\bar{x}, N)$,
- (3) the Picard sequence $\{x_n(t)\}$, defined by

$$x_{n+1}(t) = P_{2m-1}(t) + \int_0^1 |G_m(t, s)| f(s, x_n(s), x'_n(s), \dots, x_n^{(q)}(s)) ds; \quad n = 0, 1, \dots \quad (65)$$

$x_0(t) = \bar{x}(t)$ converges to $x^*(t)$ with

$$\|x^* - x_n\| \leq \theta^n N_0, \text{ and } \|x^* - x_n\| \leq \theta(1 - \theta)^{-1} \|x_n - x_{n-1}\|,$$

- (4) for any $x_0(t) = x(t) \in \bar{S}(\bar{x}, N_0)$,

$$x^*(t) = \lim_{n \rightarrow \infty} x_n(t).$$

Proof. Define an operator $T: M[0, 1] \rightarrow M[0, 1]$ as in (50). We shall show that T on $\bar{S}(\bar{x}, N)$ satisfies the conditions of Lemma 2.14. For this, if $x(t) \in \bar{S}(\bar{x}, N)$ then from the definition of norm in $M[0, 1]$ it is clear that $(x(t), x'(t), \dots, x^{(q)}(t)) \in D_3$. Further, if $x(t), y(t) \in \bar{S}(\bar{x}, N)$ then as in Theorem 3.3 it follows that

$$|(Tx)^{(2i)}(t) - (Ty)^{(2i)}(t)| \leq \theta \|x - y\| \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right]$$

and

$$|(Tx)^{(2i+1)}(t) - (Ty)^{(2i+1)}(t)| \leq \theta \|x - y\| [2 \sin \pi t + \pi(1 - 2t) \cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right].$$

The above inequalities imply that

$$\|Tx - Ty\| \leq \theta \|x - y\|.$$

Next, from (50) and (64) we have

$$(T\bar{x})(t) - \bar{x}(t) = P_{2m-1}(t) - \bar{P}_{2m-1}(t) - \int_0^1 |G_m(t, s)| \eta(s) ds$$

and hence, from the definition of approximate solution and the inequalities (40), (45) we easily find

$$\begin{aligned} |(T\bar{x})^{(2i)}(t) - \bar{x}^{(2i)}(t)| &\leq \epsilon \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t + \delta \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t \\ &\leq (\epsilon + \delta) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right] \end{aligned}$$

and

$$|(T\bar{x})^{(2i+1)}(t) - \bar{x}^{(2i+1)}(t)| \leq (\epsilon + \delta) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1 - 2t) \cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right].$$

Combining the above inequalities, to obtain

$$\|T\bar{x} - \bar{x}\| \leq (\epsilon + \delta) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right)$$

and hence

$$(1 - \theta)^{-1} \|T\bar{x} - \bar{x}\| \leq (1 - \theta)^{-1} (\epsilon + \delta) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) = N_0 \leq N.$$

Thus the conditions of Lemma 2.14 are satisfied and the conclusions (1)–(4) follow.

5. APPROXIMATE PICARD'S ITERATES

In Theorem 4.1 the conclusion (3) ensures that the sequence $\{x_n(t)\}$ obtained from (65) converges to the solution $x^*(t)$ of the boundary value problem (1), (2). However, in practical evaluation this sequence is approximated by the computed sequence, say, $\{y_n(t)\}$. To find $y_{n+1}(t)$, the function f is approximated by f_n . Therefore, the computed sequence $\{y_n(t)\}$ satisfies the recurrence relation

$$y_{n+1}(t) = P_{2m-1}(t) + \int_0^1 |G_m(t, s)| f_n(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) ds;$$

$$n = 0, 1, \dots, y_0(t) = x_0(t) = \bar{x}(t). \quad (66)$$

With respect to f_n , we shall assume the following condition:

Condition C_1

For $y_n(t), y'_n(t), \dots, y_n^{(q)}(t)$ obtained from (66), the following inequality is satisfied:

$$|f(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t)) - f_n(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t))|$$

$$\leq \Delta |f(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t))|; \quad n = 0, 1, \dots \quad (67)$$

where Δ is a nonnegative constant.

Inequality (67) corresponds to the relative error in approximating the function f by f_n for the $(n+1)$ st iteration.

Theorem 5.1

With respect to the boundary value problem (1), (2) we assume that there exists an approximate solution $\bar{x}(t)$, and the Condition C_1 is satisfied. Further, we assume that

- (i) condition (i) of Theorem 4.1 is satisfied;
- (ii) $\theta_1 = (1 + \Delta)\theta < 1$;
- (iii)

$$N_1 = (1 - \theta_1)^{-1}(\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \leq N,$$

where

$$F = \max_{0 \leq t \leq 1} |f(t, \bar{x}(t), \bar{x}'(t), \dots, \bar{x}^{(q)}(t))|.$$

Then, the following hold

- (1) all the conclusions (1)–(4) of Theorem 4.1 hold;
- (2) the sequence $\{y_n(t)\}$ obtained from (66) remains in $\bar{S}(\bar{x}, N_1)$;
- (3) the sequence $\{y_n(t)\}$ converges to $x^*(t)$, the solution of (1), (2) if and only if

$$\lim_{n \rightarrow \infty} a_n = 0,$$

where

$$a_n = \left\| y_{n+1}(t) - P_{2m-1}(t) - \int_0^1 |G_m(t, s)| f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) ds \right\| \quad (68)$$

and the following error estimate holds:

$$\|x^* - y_{n+1}\| \leq (1 - \theta)^{-1} \left[\theta \|y_{n+1} - y_n\| + \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \Delta \max_{0 \leq t \leq 1} |f(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t))| \right]. \quad (69)$$

Proof. Since $\theta_1 < 1$ implies $\theta < 1$ and obviously $N_0 \leq N_1$, the conditions of Theorem 4.1 are satisfied and conclusion (1) follows.

To prove (2), we note that $\bar{x}(t) \in \bar{S}(\bar{x}, N_1)$, and from (64) and (66) we find

$$y_1(t) - \bar{x}(t) = P_{2m-1}(t) - \bar{P}_{2m-1}(t) + \int_0^1 |G_m(t, s)| [f_0(s, \bar{x}(s), \bar{x}'(s), \dots, \bar{x}^{(q)}(s)) - f(s, \bar{x}(s), \bar{x}'(s), \dots, \bar{x}^{(q)}(s)) - \eta(s)] ds.$$

Thus, as earlier in Theorem 4.1 it follows that

$$\begin{aligned} |y^{(2i)}(t) - \bar{x}^{(2i)}(t)| &\leq (\epsilon + \delta) \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t + \Delta F \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t \\ &\leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \sin \pi t, \quad 0 \leq i \leq \left[\frac{q}{2} \right] \end{aligned} \quad (70)$$

and

$$\begin{aligned} |y^{(2i+1)}(t) - \bar{x}^{(2i+1)}(t)| &\leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1-2t)\cos \pi t], \quad 0 \leq i \leq \left[\frac{q-1}{2} \right]. \end{aligned} \quad (71)$$

From (70) and (71), we have

$$\|y_1 - \bar{x}\| \leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \leq N_1,$$

i.e. $y_1(t) \in \bar{S}(\bar{x}, N_1)$.

Now we shall assume that $y_n(t) \in \bar{S}(\bar{x}, N_1)$ and show that $y_{n+1}(t) \in \bar{S}(\bar{x}, N_1)$. For this, from (64) and (66) we have

$$y_{n+1}(t) - \bar{x}(t) = P_{2m-1}(t) - \bar{P}_{2m-1}(t) + \int_0^1 |G_m(t, s)| [f_n(s, y_n(s), y_n'(s), \dots, y_n^{(q)}(s)) - f(s, \bar{x}(s), \bar{x}'(s), \dots, \bar{x}^{(q)}(s)) - \eta(s)] ds$$

and hence it follows that

$$\begin{aligned} |y_{n+1}^{(2i)}(t) - \bar{x}^{(2i)}(t)| &\leq (\epsilon + \delta) \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t + \int_0^1 |G_{m-i}(t, s)| [|f_n(s, y_n(s), y_n'(s), \dots, y_n^{(q)}(s)) - f(s, y_n(s), y_n'(s), \dots, y_n^{(q)}(s))| \\ &\quad + |f(s, y_n(s), y_n'(s), \dots, y_n^{(q)}(s)) - f(s, \bar{x}(s), \bar{x}'(s), \dots, \bar{x}^{(q)}(s))|] ds \\ &\leq (\epsilon + \delta) \frac{1}{\pi^{2m-2i-2}} \left(\frac{1}{2\pi} \right) \sin \pi t \\ &\quad + \int_0^1 |G_{m-i}(t, s)| \left(\Delta F + (1 + \Delta) \sum_{i=0}^q L_i |y_n^{(i)}(s) - \bar{x}^{(i)}(s)| \right) ds \\ &\leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) \sin \pi t + \theta_1 \|y_n - \bar{x}\| \sin \pi t, \end{aligned} \quad (72)$$

and similarly

$$\begin{aligned} |y_{n+1}^{(2i+1)}(t) - \bar{x}^{(2i+1)}(t)| &\leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi} \right) [2 \sin \pi t + \pi(1-2t)\cos \pi t] \\ &\quad + \theta_1 \|y_n - \bar{x}\| [2 \sin \pi t + \pi(1-2t)\cos \pi t]. \end{aligned} \quad (73)$$

Combining (72) and (73) and using the fact that $y_n(t) \in \bar{S}(\bar{x}, N_1)$, it follows that

$$\begin{aligned}\|y_{n+1} - \bar{x}\| &\leq (\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) + \theta_1(1 - \theta_1)^{-1}(\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) \\ &= (1 - \theta_1)^{-1}(\epsilon + \delta + \Delta F) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) = N_1\end{aligned}$$

and hence $y_{n+1}(t) \in \bar{S}(\bar{x}, N_1)$. This completes the proof of (2).

Next, from the definition of $x_{n+1}(t)$ and $y_{n+1}(t)$, we have

$$\begin{aligned}x_{n+1}(t) - y_{n+1}(t) &= P_{2m-1}(t) + \int_0^1 |G_m(t, s)| f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) \, ds - y_{n+1}(t) \\ &\quad + \int_0^1 |G_m(t, s)| [f(s, x_n(s), x'_n(s), \dots, x_n^{(q)}(s)) - f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s))] \, ds\end{aligned}$$

and hence, as earlier it is easy to obtain

$$\begin{aligned}\|x_{n+1} - y_{n+1}\| &\leq \|y_{n+1}(t) - P_{2m-1}(t) - \int_0^1 |G_m(t, s)| f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) \, ds\| + \theta \|x_n - y_n\| \\ &= a_n + \theta \|x_n - y_n\|.\end{aligned}$$

Since $x_0(t) = y_0(t) = \bar{x}(t)$, the above inequality gives

$$\|x_{n+1} - y_{n+1}\| \leq \sum_{i=0}^n \theta^{n-i} a_i. \quad (74)$$

Now using (74) in the triangle inequality, we get

$$\|x^* - y_{n+1}\| \leq \sum_{i=0}^n \theta^{n-i} a_i + \|x_{n+1} - x^*\|. \quad (75)$$

In (75), Theorem 4.1 ensures that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| = 0.$$

Thus,

$$\lim_{n \rightarrow \infty} a_n = 0$$

is necessary and sufficient for the convergence of the sequence $\{y_n(t)\}$ to $x^*(t)$ follows from Toeplitz' lemma "for any $0 \leq \alpha < 1$, let

$$s_n = \sum_{i=0}^n \alpha^{n-i} d_i;$$

$n = 0, 1, \dots$, then

$$\lim_{n \rightarrow \infty} s_n = 0$$

if and only if

$$\lim_{n \rightarrow \infty} d_n = 0."$$

Finally, to prove (69) we note that

$$\begin{aligned}x^*(t) - y_{n+1}(t) &= \int_0^1 |G_m(t, s)| [f(s, x^*(s), x^{*'}(s), \dots, x^{*(q)}(s)) - f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) \\ &\quad + f(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s)) - f_n(s, y_n(s), y'_n(s), \dots, y_n^{(q)}(s))] \, ds\end{aligned}$$

and as earlier we find

$$\|x^* - y_{n+1}\| \leq \theta \|x^* - y_n\| + \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) \Delta \max_{0 \leq t \leq 1} |f(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t))|. \quad (76)$$

From (76) the inequality (69) is immediate.

In our next result we shall assume that

Condition C₂

For $y_n(t), y'_n(t), \dots, y_n^{(q)}(t)$ obtained from (66), the following inequality is satisfied:

$$|f(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t)) - f_n(t, y_n(t), y'_n(t), \dots, y_n^{(q)}(t))| \leq \nabla \quad (77)$$

where ∇ is a nonnegative constant.

Inequality (77) corresponds to an absolute error in approximating the function f by f_n for the $(n+1)$ st iteration.

Theorem 5.2

With respect to the boundary value problem (1), (2) we assume that there exists an approximate solution $\bar{x}(t)$, and the Condition C₂ is satisfied. Further, we assume that

- (i) condition (i) of Theorem 4.1 is satisfied;
- (ii)

$$N_2 = (1 - \theta)^{-1}(\epsilon + \delta + \nabla) \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) \leq N.$$

Then, the following hold

- (1) all the conclusions (1)–(4) of Theorem 4.1 hold;
- (2) the sequence $\{y_n(t)\}$ obtained from (66) remains in $\mathcal{S}(\bar{x}, N_2)$;
- (3) the sequence $\{y_n(t)\}$ converges to $x^*(t)$, the solution of (1), (2) if and only if

$$\lim_{n \rightarrow \infty} a_n = 0,$$

and the following error estimate holds

$$\|x^* - y_{n+1}\| \leq (1 - \theta)^{-1} \left[\theta \|y_{n+1} - y_n\| + \frac{1}{\pi^{2m-q-2}} \left(\frac{1}{2\pi}\right) \nabla \right].$$

Proof. The proof is similar to that of Theorem 5.1.

6. MONOTONE CONVERGENCE

It is well recognized that the method of upper and lower solutions, together with uniformly monotone convergent technique offers effective tools in proving and constructing multiple solutions of nonlinear problems. The upper and lower solutions that generate an interval in a suitable partially ordered space serve as upper and lower bounds for solutions which can be improved by uniformly monotone convergent iterative procedures. Obviously, from the computational point of view monotone convergence has superiority over ordinary convergence and several monotonic iterative schemes for ordinary and partial differential equations have been developed and analysed in Ref. [21], which covers more than 100 recent publications on the subject, and papers [22, 23]. Here we shall extend this fruitful technology for the boundary value problem (1), (2) with $q = 0$. For this, we need the existence of a lower and an upper solution of (1), (2) with $q = 0$, which are defined as follows: we call a function $\mu(t) \in C^{(2m)}[0, 1]$ a lower solution of (1), (2) with $q = 0$ provided

$$(-1)^m \mu^{(2m)}(t) \leq f(t, \mu(t)), \quad t \in [0, 1];$$

$$(-1)^i [\mu^{(2i)}(0) - \alpha_i] \leq 0, \quad (-1)^i [\mu^{(2i)}(1) - \beta_i] \leq 0; \quad i = 0, 1, \dots, m-1.$$

Similarly, a function $v(t) \in C^{(2m)}[0, 1]$ is called an upper solution of (1), (2) with $q = 0$ if

$$\begin{aligned} (-1)^m v^{(2m)}(t) &\geq f(t, v(t)), \quad t \in [0, 1]; \\ (-1)^i [v^{(2i)}(0) - \alpha_i] &\geq 0, \quad (-1)^i [v^{(2i)}(1) - \beta_i] \geq 0; \quad i = 0, 1, \dots, m-1. \end{aligned}$$

Lemma 6.1

Let $\mu(t)$ and $v(t)$ be lower and upper solutions of (1), (2) with $q = 0$, and $P_{2m-1,\mu}(t)$ and $P_{2m-1,v}(t)$ be the polynomials of degree $(2m-1)$ satisfying

$$P_{2m-1,\mu}^{(2i)}(0) = \mu^{(2i)}(0), \quad P_{2m-1,\mu}^{(2i)}(1) = \mu^{(2i)}(1); \quad i = 0, 1, \dots, m-1$$

and

$$P_{2m-1,v}^{(2i)}(0) = v^{(2i)}(0), \quad P_{2m-1,v}^{(2i)}(1) = v^{(2i)}(1); \quad i = 0, 1, \dots, m-1,$$

respectively. Then, for all $t \in [0, 1]$

$$P_{2m-1,\mu}(t) \leq P_{2m-1}(t) \leq P_{2m-1,v}(t).$$

Proof. Explicitly $P_{2m-1,\mu}(t)$ can be written as

$$\begin{aligned} P_{2m-1,\mu}(t) &= \sum_{i=0}^{m-1} [\mu^{(2i)}(0)A_i(1-t) + \mu^{(2i)}(1)A_i(t)] \\ &= \sum_{i=0}^{m-1} [(-1)^i \mu^{(2i)}(0)(-1)^i A_i(1-t) + (-1)^i \mu^{(2i)}(1)(-1)^i A_i(t)]. \end{aligned}$$

Now since $(-1)^i A_i(t)$ as well as $(-1)^i A_i(1-t)$ are nonnegative for all $t \in [0, 1]$, from the definition of $\mu(t)$ it follows that

$$\begin{aligned} P_{2m-1,\mu}(t) &\leq \sum_{i=0}^{m-1} [(-1)^i \alpha_i (-1)^i A_i(1-t) + (-1)^i \beta_i (-1)^i A_i(t)] \\ &= \sum_{i=0}^{m-1} [\alpha_i A_i(1-t) + \beta_i A_i(t)] \\ &= P_{2m-1}(t). \end{aligned}$$

The proof of the inequality $P_{2m-1}(t) \leq P_{2m-1,v}(t)$ is similar.

In the space $C[0, 1]$ we shall consider the norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|,$$

and introduce a partial ordering as follows: for $x, y \in C[0, 1]$ we say that $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0, 1]$.

Theorem 6.2

With respect to the boundary value problem (1), (2) with $q = 0$ we assume that $f(t, u_0)$ is nondecreasing in u_0 . Further, let there exist lower and upper solutions $x_0(t), y_0(t)$ such that $x_0 \leq y_0$. Then, the sequences $\{x_n\}, \{y_n\}$ where $x_n(t)$ and $y_n(t)$ are defined by the iterative schemes

$$\begin{aligned} x_{n+1}(t) &= P_{2m-1}(t) + \int_0^1 G_m(t, s) |f(s, x_n(s))| ds \\ y_{n+1}(t) &= P_{2m-1}(t) + \int_0^1 G_m(t, s) |f(s, y_n(s))| ds \end{aligned}$$

are well defined, and $\{x_n\}$ converges to an element $x \in C[0, 1]$, $\{y_n\}$ converges to an element $y \in C[0, 1]$ (the convergence being in the norm of $C[0, 1]$). Further $x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq x \leq y \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0$, $x(t), y(t)$ are solutions of (1), (2) with $q = 0$, and each solution $z(t)$ of this problem which is such that $z \in [x_0, y_0]$ satisfies $x \leq z \leq y$.

Proof. First, we shall show that the operator $T: C[0, 1] \rightarrow C[0, 1]$ defined by

$$Tx(t) = P_{2m-1}(t) + \int_0^1 |G_m(t, s)| f(s, x(s)) \, ds$$

is isotone. For this, let $x, y \in C[0, 1]$ and $x \leq y$, then from the partial ordering it follows that $x(s) \leq y(s)$ for all $s \in [0, 1]$, and hence from the monotonic property of f , we have $f(s, x(s)) \leq f(s, y(s))$, $s \in [0, 1]$. Next, since $(-1)^m G_m(t, s) = |G_m(t, s)|$ it follows that

$$|G_m(t, s)| f(s, x(s)) \leq |G_m(t, s)| f(s, y(s)); \quad s, t \in [0, 1].$$

From this the inequality $T(x) \leq T(y)$ is obvious, and this completes the proof of T being isotone.

Next, since $x_0(t)$ is a lower solution, Lemma 6.1 gives that

$$\begin{aligned} x_0(t) &= P_{2m-1, x_0}(t) + \int_0^1 |G_m(t, s)| (-1)^m x_0^{(2m)}(s) \, ds \\ &\leq P_{2m-1}(t) + \int_0^1 |G_m(t, s)| f(s, x_0(s)) \, ds \\ &= Tx_0(t), \end{aligned}$$

i.e. $x_0 \leq T(x_0)$. The inequality $T(y_0) \leq y_0$ can be proved analogously. Thus, the conditions (i) and (i)' of Lemma 2.15 hold and in conclusion the sequences $\{T^n(x_0)\}$, $\{T^n(y_0)\}$ are well defined.

Since $T^n(x_0) = T[T^{n-1}(x_0)]$, we have $T^n(x_0) = x_n$ and $T^n(y_0) = y_n$. The sequence $\{x_n(t)\}$ is nondecreasing and bounded from above by $y_0(t)$, $t \in [0, 1]$. Similarly, the sequence $\{y_n(t)\}$ is nonincreasing and bounded from below by $x_0(t)$, $t \in [0, 1]$. Hence, in conclusion the sequences $\{x_n(t)\}$, $\{y_n(t)\}$ are uniformly bounded on $[0, 1]$. Further, since the functions $x_n(t)$, $y_n(t)$ are the solutions of appropriate boundary value problems, these sequences are equicontinuous also. Thus, Arzela-Ascoli theorem is applicable and there exist subsequences of $\{x_n(t)\}$, $\{y_n(t)\}$ which converge uniformly on $[0, 1]$. However, since these sequences are monotonic, we conclude that the whole sequences $\{x_n(t)\}$, $\{y_n(t)\}$ converge uniformly to some $x(t)$, $y(t)$, and $x \leq y$ in the partial ordering of $C[0, 1]$. Summarizing these arguments, we find that $T^n x_0 \uparrow x$ and $T^n y_0 \downarrow y$.

Finally, the continuity of the operator T implies that $T^{n+1}x_0 = T[T^n x_0] \uparrow Tx$ and $T^{n+1}y_0 = T[T^n y_0] \downarrow Ty$.

Hence the conditions of Lemma 2.15 are satisfied and the conclusions of Theorem 6.2 follow.

Remark 6.1

If $f(t, u_0)$ is nondecreasing in u_0 , then uniqueness of the solutions of (1), (2) with $q = 0$ is not guaranteed, e.g. the boundary value problem $(-1)^m x^{(2m)} = \pi^{2m} x$, (62) has an infinite number of solutions $x(t) = c \sin \pi t$, where c is an arbitrary constant. However, if $f(t, u_0)$ is nonincreasing then the problem (1), (2) with $q = 0$ has at most one solution. To prove this, we assume that $x(t)$ and $y(t)$ both are the solutions of (1), (2) so that

$$(-1)^m [x^{(2m)}(t) - y^{(2m)}(t)] = f(t, x(t)) - f(t, y(t))$$

and hence

$$(-1)^m [x(t) - y(t)][x^{(2m)}(t) - y^{(2m)}(t)] = [x(t) - y(t)][f(t, x(t)) - f(t, y(t))] \leq 0,$$

where the inequality follows as a consequence of nonincreasing nature of $f(t, x_0)$ in x_0 . Now an integration by parts gives that

$$(-1)^m \int_0^1 [x(t) - y(t)][x^{(2m)}(t) - y^{(2m)}(t)] \, dt = (-1)^{2m} \int_0^1 (x^{(m)}(t) - y^{(m)}(t))^2 \, dt \leq 0,$$

which is possible only when $x(t) \equiv y(t)$.

Theorem 6.3

The linear differential equation

$$(-1)^m x^{(2m)} = f(t)x + g(t) \quad (78)$$

together with the boundary conditions (2) has a unique solution if

$$\max_{0 \leq t \leq 1} f(t) \leq 0.$$

Proof. For the linear problem (78), (2) obviously the uniqueness implies the existence. Hence, if $f(t) \leq 0$ for all $t \in [0, 1]$ our Remark 6.1 is applicable and the result follows.

Remark 6.2

From Theorem 6.3 the differential equation

$$(-1)^m x^{(2m)} = -\pi^{2m} x + 2\pi^{2m} \sin \pi t,$$

together with the boundary conditions (62) has a unique solution $x(t) = \sin \pi t$. However, the iterative procedure given by

$$\begin{aligned} (-1)^m x_{n+1}^{(2m)} &= -\pi^{2m} x_n + 2\pi^{2m} \sin \pi t; \\ x_{n+1}^{(2i)}(0) &= x_{n+1}^{(2i)}(1) = 0, \quad 0 \leq i \leq m-1, \end{aligned}$$

with $x_0(t) = 0$ oscillates [$x_{2n+1}(t) = 2 \sin \pi t$, $x_{2n}(t) = 0$], and hence Theorem 6.3 does not imply the convergence of the iterative scheme (65) with $q = 0$.

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